

SMOOTH QUOTIENTS OF BI-ELLIPTIC SURFACES

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ABSTRACT. We consider the quotient X of bi-elliptic surface by a finite automorphism group. If X is smooth, then it is a bi-elliptic surface or ruled surface with irregularity one. As a corollary any bi-elliptic surface cannot be Galois covering of projective plane, hence does not have any Galois embedding.

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1. STATEMENT OF RESULT

We consider a covering of surface, i.e., let X_1 and X_2 be connected normal complex surfaces and $\pi : X_1 \rightarrow X_2$ a finite surjective proper holomorphic map. There are lots of studies of the covering. We are interested in the following cases (1) and (2):

- (1) X_2 is a rational surface, in particular the projective plane \mathbb{P}^2 , for example [1]
- (2) X_1 has Kodaira dimension 0, for example [3, 6]

In this note we consider the case where X_1 is a bi-elliptic surface and π is a Galois covering. The definition of bi-elliptic surface is as follows, which has been called a hyperelliptic surface [4].

Definition 1. A bi-elliptic surface is a surface with the geometric genus zero and having an abelian surface as its unramified covering.

First we note the following:

Remark 2. Let S be a bi-elliptic surface. If $\pi : S \rightarrow X$ is a Galois covering and X is smooth, then X has no curve with negative self-intersection number. In particular, if X is rational, then it is \mathbb{P}^2 or $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Suppose the contrary. Then, there exists an irreducible curve C in X such that $C^2 < 0$. We have that $(\pi^*(C))^2 = \deg \pi \cdot C^2 < 0$. Thus there exists an irreducible component C' in $\pi^*(C)$ with $C'^2 < 0$. which is a contradiction. Because there exists no curve with negative self-intersection number in bi-elliptic surface. \square

Our result is stated as follows:

Theorem 3. *Let S be a bi-elliptic surface and G a finite subgroup of $\text{Aut}(S)$, the automorphism group of S . Let $X = S/G$ be the quotient surface. If X is smooth, then it is a bi-elliptic surface or a ruled surface with irregularity one.*

As a corollary we have the following.

Corollary 4. *Bi-elliptic surface cannot be a Galois covering of any smooth rational surface.*

Miranda [2] presents a construction of bi-elliptic surface as covering of rational elliptic surface. We present the other examples.

Example 5. Let A be the abelian surface with the period matrix

$$\begin{pmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & e_n \end{pmatrix},$$

where $\Im \lambda > 0$ and $e_n = \exp(2\pi\sqrt{-1}/n)$, ($n = 4, 6$). Let σ_i ($i = 1, 2$) be the automorphism of A induced by

$$\sigma_1 z = \begin{pmatrix} 1 & 0 \\ 0 & e_n \end{pmatrix} z + \begin{pmatrix} 1/n \\ 0 \end{pmatrix} \text{ and } \sigma_2 z = \begin{pmatrix} 1 & 0 \\ 0 & e_n \end{pmatrix} z,$$

where $z \in \mathbb{C}^2$. Then $S = A/\langle \sigma_1^m \rangle$ ($n = 2m$) and $X = A/\langle \sigma_1 \rangle$ are bi-elliptic surfaces. Note that $\pi : S \rightarrow X$ is a double covering. On the other hand, we have $\sigma_1^m \sigma_2 = \sigma_2 \sigma_1^m$, hence σ_2 induces an automorphism $\bar{\sigma}_2$ on S . It is easy to see that $S/\langle \bar{\sigma}_2 \rangle$ is isomorphic to $E \times \mathbb{P}^1$, where E is an elliptic curve with the period matrix $(1/2, \lambda)$.

Remark 6. [5, Example 2.4] Let A_i ($i = 1, 2$) be the abelian surface defined by the following period matrix:

$$\Omega_1 = \begin{pmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & 0 & \omega \end{pmatrix} \text{ and } \Omega_2 = \begin{pmatrix} 1 & 0 & (\omega - 1)/3 & 0 \\ 0 & 1 & (\omega - 1)/3 & \omega \end{pmatrix}, \text{ respectively,}$$

where $\omega = \exp(2\pi\sqrt{-1}/3)$. Let σ_i be the automorphism of A_i defined by

$$\sigma_1 z = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} z + \begin{pmatrix} (\omega + 2)/3 \\ 0 \end{pmatrix} \text{ and } \sigma_2 z = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} z + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$

Then $\sigma_i^3 = id$ on A_i and $S_i = A_i/\sigma_i$ is a bi-elliptic surface. Moreover letting

$$\tau_1 z = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} z \text{ and } \sigma_2 z = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} z + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}$$

we see that τ_i defines an automorphism of S_i and S_i/τ_i turns out to be a rational surface. Note that the rational surfaces in these examples have singular points.

We have defined Galois embedding of algebraic variety and applied it to study several varieties. In particular we have shown that many abelian surfaces have Galois embeddings [6]. However, as a corollary of Theorem 3, we have the following.

Corollary 7. *There does not exist any Galois embedding of bi-elliptic surface.*

2. PROOF

Let S be a bi-elliptic surface and $\pi_1 : S \rightarrow X$ a finite Galois covering. Then S can be expressed as \mathbb{C}^2/Γ_1 , where Γ_1 is the complex crystallographic group given in [4, Theorem]. Let $A = \mathbb{C}^2/\Gamma_0$ be the abelian surface, where $\pi_2 : A \rightarrow S$ is an unramified covering such that Γ_0 is the normal subgroup of Γ_1 and $|\Gamma_1 : \Gamma_0| = 2, 3, 4$ or 6 . Put $n = |\Gamma_1 : \Gamma_0|$. Let G be the finite subgroup of $\text{Aut}(S)$ such that $X = S/G$. For $g \in G$, we let \tilde{g} be the lift of g on the universal covering \mathbb{C}^2 .

Claim 8. *The \tilde{g} turns out to be an affine transformation.*

Proof. For each $s \in \Gamma_0^n$, there exists $t \in \Gamma_0$ such that $s = t^n$. Since $\Gamma_0 \subset \Gamma_1$ and Γ_1 is a discrete group, we have $u \in \Gamma_1$ such that $\tilde{g}t = u\tilde{g}$. Hence we get $\tilde{g}t^n = u^n\tilde{g}$ and $u^n \in \Gamma_0$. This implies $\tilde{g}s \equiv \tilde{g} \pmod{\Gamma_0}$. Let \mathcal{L}_0 be the lattice defining the abelian surface $A = \mathbb{C}^2/\Gamma_0$. Putting $z = {}^t(z_1, z_2) \in \mathbb{C}^2$, $\tilde{g}(z_1, z_2) = (\tilde{g}_1(z_1, z_2), \tilde{g}_2(z_1, z_2))$, we have

$$\tilde{g}_i(z_1 + \alpha_1, z_2 + \alpha_2) = \tilde{g}_i(z_1, z_2) + \beta_i,$$

where $i = 1, 2$, ${}^t(\alpha_1, \alpha_2) \in n\mathcal{L}_0$ and ${}^t(\beta_1, \beta_2) \in \mathcal{L}_0$. Hence we get

$$s^* \left(\frac{\partial \tilde{g}_i}{\partial z_j} \right) = \frac{\partial \tilde{g}_i}{\partial z_j} \quad (i, j = 1, 2).$$

This means $\partial \tilde{g}_i / \partial z_j$ is a holomorphic function on \mathbb{C}^2/Γ_0^n , which is also an abelian surface. Therefore $\partial \tilde{g}_i / \partial z_j$ is a constant, i.e., \tilde{g} is an affine transformation. \square

Thus \tilde{g} has the representation $\tilde{g}z = M(g)z + v(g)$, where $z \in \mathbb{C}^2$, $M(g) \in GL(2, \mathbb{C})$ and $v(g) \in \mathbb{C}^2$. We use this expression hereafter. Let Γ_2 be the affine transformation group generated by $\{\tilde{g} \mid g \in G\}$ and Γ_1 . Then we have $X = \mathbb{C}^2/\Gamma_2$. Since $\pi_1 : S \rightarrow X$ is a Galois covering, we have Γ_0 (resp. Γ_1) is a normal subgroup of Γ_1 (resp. Γ_2) and $\Gamma_2/\Gamma_1 \cong G$. Let Γ_1/Γ_0 be generated by σ . Then, we have the expression $\tilde{\sigma}z = M(\sigma)z + v(\sigma)$, where

$$M(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & e_n \end{pmatrix} \quad \text{and} \quad v(\sigma) = \begin{pmatrix} 1/n \\ 0 \end{pmatrix}.$$

Claim 9. *The X has two fibrations.*

Proof. Since Γ_1 is a normal subgroup of Γ_2 , there exists an integer r such that $\tilde{g}\tilde{\sigma}\tilde{g}^{-1} = t\tilde{\sigma}^r$, where $g \in G$ and $t \in \Gamma_0$. This means $M(g)M(\sigma)M(g)^{-1} = M(\sigma)^r$, hence $r = 1$. Therefore we have $M(g)M(\sigma) = M(\sigma)M(g)$. If we put $M(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it is easy to see that $M(g)$ is a diagonal matrix. Hence the lift of each element γ of Γ_2 can be expressed as

$$\tilde{\gamma}(z) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} z + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (*)$$

Since \mathcal{L}_0 is the lattice defining the abelian surface A , we have $\Gamma_0 = \{\ell \mid \ell(z) = z + {}^t(b_1, b_2), \text{ where } {}^t(b_1, b_2) \in \mathcal{L}_0\}$.

Put $\Gamma_{0i} = \{\ell_i \mid \ell_i(z_i) = z_i + b_i, \text{ where } {}^t(b_1, b_2) \in \mathcal{L}_0\}$. Referring to the list of abelian surfaces in [4], we infer that \mathbb{C}/Γ_{0i} is an elliptic curve. Using the above representation (*), we define

$$\Gamma_{2i} = \{\gamma_i \mid \gamma_i(z_i) = \alpha_i z_i + a_i, \text{ where } \gamma \in \widetilde{\Gamma_2}\}.$$

Then Γ_{0i} is a subgroup of Γ_{2i} with a finite index. Hence Γ_{0i} is a discrete subgroup and \mathbb{C}/Γ_{2i} is a smooth curve. Thus we have the following diagram

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{p_i} & \mathbb{C} \\ \pi \downarrow & & \pi_i \downarrow \\ X = \mathbb{C}^2/\Gamma_2 & \xrightarrow{\bar{p}_i} & C_i = \mathbb{C}/\Gamma_{2i} \end{array},$$

where $i = 1, 2$. Consequently we get two fibrations $\bar{p}_i : X \longrightarrow C_i$. \square

Suppose that X is smooth. Then we have

$$\dim H^1(X, \mathcal{O}_X) \leq \dim H^1(S, \mathcal{O}_S) = 1, \text{ and}$$

$$\dim H^2(X, \mathcal{O}_X) \leq \dim H^2(S, \mathcal{O}_S) = 0$$

and the Kodaira dimension of X is less than or equal to 0. By Remark 2 and the classification theorem of algebraic surfaces, we infer that X is $\mathbb{P}^1 \times \mathbb{P}^1$, a ruled surface with irregularity one or a bi-elliptic surface.

Claim 10. *The X cannot be $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. In the representation above $(*)$ there exists i such that $\alpha_i \neq 1$. Then the fiber space $\bar{p}_i : X \longrightarrow C_i$ has a multiple singular fiber. Let F_i be the fiber of the projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. Then each divisor D is linearly equivalent to $n_1 F_1 + n_2 F_2$. If $D^2 = 0$, then we have $n_1 = 0$ or $n_2 = 0$. So that there are only two fibrations from $\mathbb{P}^1 \times \mathbb{P}^1$ to \mathbb{P}^1 , which are the first and second projections. Neither of projections have any multiple fiber. Combining the above assertions we infer readily the conclusion. \square

This completes the proof.

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